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LETTER TO THE EDITOR

Topology of non-singular textures of superfluid ^{3}He

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Abstract. The topological classification of non-singular textures of superfluid ^{3}He is discussed using relative homotopy groups.

Topological homotopy group methods have been introduced to classify point and line singularities in superfluid ^{3}He by Toulouse and Kleman (1976) and Volovik and Mineev (1977a, b). In the case of line singularities, the idea is to discover the topologically inequivalent mappings from a circle encircling the singular line to order parameter space. If the superfluid is contained in a cylindrical vessel one might imagine instead mapping a cross section of the cylinder into order parameter space, rather than a single circle within that cross section. This might provide a more complete classification of non-singular textures though, of course, it would not make sense for singular textures, because the mapping would be discontinuous on the central singular line. It is this question that we discuss.

In general, let the space of order parameters be R. The interior of the disc representing the cross section of the cylindrical container is mapped into R. If there are boundary conditions, the boundary of the disc will be mapped into some subspace A of R. Usually, the boundary of the disc will be mapped into some specific class of curves in A. For example, if the boundary conditions are $\boldsymbol{l}=\hat{\boldsymbol{\rho}}$ (the unit radial vector in cylindrical polar coordinates) then A is formed by restricting the $\boldsymbol{l}$ vector to be perpendicular to the axis of the cylindrical container. However, at the end of the day, we may further restrict attention to curves in A with unit winding for $\boldsymbol{l}$ around the boundary.

We proceed by studying the relative homotopy group $\pi_2(R, A)$. (See, for example, Steenrod 1951.) At the end we restrict attention to those elements of $\pi_2(R, A)$ which match the full boundary conditions. Information about the relative homotopy group may be obtained from the exact sequence (in which the image of one homomorphism is the kernel of the next)

$$\pi_2(A) \xrightarrow{\alpha} \pi_2(R) \xrightarrow{\beta} \pi_2(R, A) \xrightarrow{\gamma} \pi_1(A) \xrightarrow{\delta} \pi_1(R) \qquad (1)$$

where α, β and δ are induced by inclusion mappings, and γ by a boundary mapping. Because of the exactness of the sequence

$$\gamma\pi_2(R, A) = \ker \delta. \qquad (2)$$

Thus all elements of $\pi_2(R, A)$ have boundaries associated with the identity element of $\pi_1(R)$. This ensures that we are dealing with non-singular textures.

In the absence of boundary conditions, $A = R$, and the mappings α and δ are identity mappings. The exactness of the homotopy sequence then leads to

$$\pi_2(R, R) = 0. \tag{3}$$

Thus without boundary conditions nothing can be said about non-singular textures other than that they correspond to the identity of $\pi_1(R)$. We therefore restrict attention to cases where boundary conditions are to be applied.

If $\pi_2(R) = 0$, then

$$\pi_2(R, A) \cong \gamma\pi_2(R, A) \subset \pi_1(A), \tag{4}$$

and no more can be learned about non-singular textures by considering $\pi_2(R, A)$ than can be learned by considering $\pi_1(A)$, i.e. by considering the texture on the boundary.

If $\pi_2(R) \neq 0$, then two cases of particular interest arise. The first is when $\pi_2(A) = 0$. This will often occur because the boundary conditions on a cylinder may spoil spherical symmetry. In this case, $\ker \beta = 0$, and the homotopy sequence leads to

$$\pi_2(R, A)/\pi_2(R) \cong \ker \delta. \tag{5}$$

Consequently, more can be learned from mapping the complete disc into order parameter space than can be learned from mapping the boundary curve, i.e. from $\pi_2(R, A)$ than from $\ker \delta$.

A physical example of this case is the (dipole-locked) B phase in cylinders of radius at least 1 cm (so that the boundary free energy is greater than the bending free energy and the boundary conditions may be applied). Apart from a phase factor which is uninteresting for our present purposes, the order parameter space R is a space of three-dimensional unit vectors $\boldsymbol{n}$. Thus,

$$R = S^2 \tag{6}$$

(Volovik and Mineev 1977a).

The boundary condition $\boldsymbol{n} = \hat{\boldsymbol{\rho}}$ restricts $\boldsymbol{n}$ to be in a space perpendicular to the axis of the cylinder, so

$$A = S^1 \tag{7}$$

and the exact sequence of equation (1) leads to

$$\pi_2(R, A) = Z + Z. \tag{8}$$

It is easy to see that the two integers correspond to mappings where the whole disc is mapped m times on to the northern hemisphere of the sphere S^2, and n times on to the southern hemisphere of the sphere. Since the exact boundary condition is $\boldsymbol{n} = \hat{\boldsymbol{\rho}}$, the mappings of physical interest are those for which m and n differ by one.

Simple examples may be obtained by first continuously deforming the disc to a hemisphere. We take coordinate axes with the z axis through the north pole of the hemisphere and the x and y axes in the plane of its base. Our examples take their simplest forms in terms of spherical polar coordinates (θ', ϕ') based on the y axis rather than the usual spherical polars (θ, ϕ) based on the z axis.

Then the surface of the hemisphere may be written as

$$\boldsymbol{r} = \hat{\boldsymbol{y}} \cos \theta' + \sin \theta'(\hat{\boldsymbol{z}} \sin \phi' + \hat{\boldsymbol{x}} \cos \phi') \tag{9}$$

with

$$0 \leq \theta' \leq \pi, \qquad 0 \leq \phi' \leq \pi.$$

We define the mappings to the sphere S^2 by

$$\boldsymbol{n} = \hat{\boldsymbol{y}} \cos\theta' + \sin\theta'[\hat{\boldsymbol{z}} \sin(2N+1)\phi' + \hat{\boldsymbol{x}} \cos(2N+1)\phi']. \tag{10}$$

As the hemisphere is covered once the northern hemisphere of the sphere is covered $N+1$ times and the southern hemisphere N times. The western half of the boundary of the hemisphere is mapped on to the western half of the equator of the sphere, and similarly for the eastern half. In this way the boundary condition $\boldsymbol{n} = \hat{\boldsymbol{\rho}}$ is satisfied. In terms of the cylindrical polar coordinates (ρ, Φ) defining position on the disc from which the hemisphere was obtained by deformation, we may take

$$\cos\theta' = \frac{\rho}{R} \sin\Phi \tag{11}$$

and

$$\cos\phi' = \frac{\rho\cos\Phi}{\sqrt{(R^2 - \rho^2 \sin^2\Phi)}}, \tag{12}$$

where R is the radius of the cylindrical container. The simplest example, $N = 0$ is

$$\boldsymbol{n} = \hat{\boldsymbol{z}} \sqrt{\left(1 - \frac{\rho^2}{R^2}\right)} + \hat{\boldsymbol{\rho}} \frac{\rho}{R} \tag{13}$$

and resembles the Mermin and Ho texture for the A phase (Mermin and Ho 1976). There are, of course, topologically inequivalent textures in which the roles of the northern and southern hemispheres are interchanged ($\hat{\boldsymbol{z}} \to -\hat{\boldsymbol{z}}$).

Another example of this type is the academic case of (metastable) dipole-free ^{3}He–A$_1$ in zero magnetic field. Then $R = (SO_3 \times SO_3)/S^1$ and $\pi_2(R) = Z$ (Bailin and Love 1978a); but $\boldsymbol{l} = \hat{\boldsymbol{\rho}}$ gives $A = S^1 \times SO_3$, so that $\pi_2(A) = 0$. It follows that the textures satisfying the exact boundary condition are again labelled by an integer.

The other case of physical interest with $\pi_2(R) \neq 0$ is $\pi_2(A) = \pi_2(R)$. This can arise when the boundary conditions do not couple to the degrees of freedom which can produce point singularities. Then, the mapping α in equation (1) is the identity mapping, and the exact sequence leads to

$$\pi_2(R, A) \cong \ker\delta. \tag{14}$$

Thus, all the information about the texture is already contained in the boundary texture.

The A phase in cylinders of radius much less than 10^{-3} cm provides a good example, because it is the $\boldsymbol{d}$ vector that can produce point singularities and the $\boldsymbol{l}$ vector that couples to the boundary conditions. From Bailin and Love 1978b, examples of elements of $\pi_1(A)$ that belong to $\ker\delta$ are

$$\boldsymbol{\Delta} = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{\Phi}} + \mathrm{i}\hat{\boldsymbol{z}}) \exp(\mathrm{i}m\phi)$$

$$\boldsymbol{d} = \hat{\boldsymbol{x}} \cos n\phi + \hat{\boldsymbol{y}} \sin n\phi \tag{15}$$

with

$$2m - 2 = 0(\mathrm{mod}\ 4).$$

$m = 1, 3, 5$ etc are all inequivalent textures which are however non-singular.

Our arguments may also be applied in spherical containers by studying $\pi_3(R, A)$ rather than $\pi_2(R, A)$.*pr However, for the cases of interest for superfluid* ^{3}*He the boundary conditions are that* $\boldsymbol{n}$ or $\boldsymbol{l}$ is normal to the surface of the spherical container, and in consequence $A = R$. Since $\pi_3(R, R) = 0$, nothing new is learned by considering the relative homotopy group (even when non-singular textures are allowed by the boundary conditions).

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Note added in proof. Since completing this work we have received a Landau Institute preprint by V P Mineev and G E Volovik, entitled 'Planar and linear solitons in superfluid ^{3}He', which contains some similar ideas but different applications.

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